

THE COX RING OF A DEL PEZZO SURFACE

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ABSTRACT. Let X_r be a smooth Del Pezzo surface obtained from \mathbb{P}^2 by blowing up $r \leq 8$ points in general position. It is well known that for $r \in \{3, 4, 5, 6, 7, 8\}$ the Picard group $\text{Pic}(X_r)$ contains a canonical root system $R_r \in \{A_2 \times A_1, A_4, D_5, E_6, E_7, E_8\}$. We prove some general properties of the Cox ring of X_r ($r \geq 4$) and show its similarity to the homogeneous coordinate ring of the orbit of the highest weight vector in some irreducible representation of the algebraic group G associated with the root system R_r .

1. INTRODUCTION

Let X be a projective algebraic variety over a field \mathbb{k} . Assume that the Picard group $\text{Pic}(X)$ is a finitely generated abelian group. Consider the vector space

$$\Gamma(X) := \bigoplus_{[D] \in \text{Pic}(X)} H^0(X, \mathcal{O}(D)).$$

One wants to make it an \mathbb{k} -algebra which is graded by the monoid of effective classes in $\text{Pic}(X)$ such that the algebra structure will be compatible with the natural bilinear map

$$b_{D_1, D_2} : H^0(X, \mathcal{O}(D_1)) \times H^0(X, \mathcal{O}(D_2)) \rightarrow H^0(X, \mathcal{O}(D_1 + D_2)).$$

However, there exist some problems in the realization of this idea. First of all there is no any natural isomorphism between $H^0(X, \mathcal{O}(D))$ and $H^0(X, \mathcal{O}(D'))$ if $[D] = [D']$. There exists only a canonical bijection between the linear systems $|D| \cong |D'|$ (where $|D|$ is the projectivization of the \mathbb{k} -vector space $H^0(X, \mathcal{O}(D))$). As a consequence, the bilinear map b_{D_1, D_2} depends not only on the classes $[D_1], [D_2], [D_1 + D_2] \in \text{Pic}(X)$, but also on their particular representatives. One can easily see that only the morphism

$$s_{[D_1], [D_2]} : |D_1| \times |D_2| \rightarrow |D_1 + D_2|$$

of the product of two projective spaces $|D_1| \times |D_2|$ to another projective space $|D_1 + D_2|$ is well-defined. For this reason, it is much more natural to consider the graded set of projective spaces

$$|\Gamma(X)| := \bigsqcup_{[D] \in \text{Pic}(X)} |D|$$

together with all possible morphisms $s_{[D_1], [D_2]}$ any two effective classes $[D_1], [D_2] \in \text{Pic}(X)$.

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Inspired by the paper of Cox on the homogeneous ring of a toric variety [Cox], Hu and Keel [H-K] suggested a definition of a *Cox ring*

$$\text{Cox}(X) = R(X, L_1, \dots, L_r) := \bigoplus_{(m_1, \dots, m_r) \in \mathbb{Z}^r} H^0(X, \mathcal{O}(m_1 L_1 + \dots + m_r L_r))$$

which uses a choice of some \mathbb{Z} -basis L_1, \dots, L_r in $\text{Pic}(X)$ (e.g. if $\text{Pic}(X) \cong \mathbb{Z}^r$ is a free abelian group). Using such a \mathbb{Z} -basis, one obtains a particular representative for each class in $\text{Pic}(X)$ together with a well-defined multiplication so $R(X, L_1, \dots, L_r)$ becomes a well-defined \mathbb{k} -algebra. If L'_1, \dots, L'_r is another \mathbb{Z} -basis of $\text{Pic}(X)$, then the corresponding Cox algebra $R(X, L'_1, \dots, L'_r)$ is isomorphic to $R(X, L_1, \dots, L_r)$. Unfortunately, we can not expect to choose a \mathbb{Z} -basis of $\text{Pic}(X)$ in a natural canonical way. More often one can choose in a natural way some effective divisors D_1, \dots, D_n on X such that $\text{Pic}(X)$ is generated by $[D_1], \dots, [D_n]$. If we set

$$U := X \setminus (D_1 \cup \dots \cup D_n)$$

and assume that X is smooth, then $\text{Pic}(U) = 0$ and we obtain the exact sequence

$$1 \rightarrow \mathbb{k}^* \rightarrow \mathbb{k}[U]^* \rightarrow \bigoplus_{i=1}^n \mathbb{Z}[D_i] \rightarrow \text{Pic}(X) \rightarrow 0.$$

Choosing a \mathbb{k} -rational point p in U , we can split the monomorphism $\mathbb{k}^* \rightarrow \mathbb{k}[U]^*$, so that one has an isomorphism

$$\mathbb{k}[U]^* \cong \mathbb{k}^* \oplus G,$$

where $G \subset \mathbb{k}[U]^*$ is a free abelian group of rank $n - r$. The choice of a \mathbb{k} -rational point $p \in U$ allows to give another approach to the graded space $\Gamma(X)$ and to the Cox algebra:

Definition 1.1. Let X, U, p, D_1, \dots, D_n be as above. We consider the graded \mathbb{k} -algebra

$$\Gamma(X, U, p) := \bigoplus_{(m_1, \dots, m_n) \in \mathbb{Z}^n} H^0(X, \mathcal{O}(m_1 D_1 + \dots + m_n D_n))$$

and define

$$\text{Cox}(X, U, p) := \Gamma(X, U, p)_G$$

as the quotient of the $\Gamma(X, U, p)$ modulo the ideal generated by

$$\{x - gx \mid x \in \Gamma(X, U, p), g \in G\}.$$

Since $\text{Pic}(X) \cong \mathbb{Z}^n / G$, we obtain a natural $\text{Pic}(X)$ -grading on $\text{Cox}(X, U, p)$.

We expect that the algebra $\text{Cox}(X, U, p)$ can be applied to some arithmetic questions on \mathbb{k} -rational points in $U \subset X$.

Remark 1.2. The above definition of the ring $\text{Cox}(X, U, p)$ depends on the choice of an open subset $U \subset X$ and a \mathbb{k} -rational point $p \in U$. A similar idea was used by Colliot-Thélène and Sansuc in [C-S] for constructing universal torsors and deriving explicit equations for them. The lack of a canonical construction is precisely what makes descending the universal torsor an interesting problem. Some applications of the universal torsor for Del Pezzo surfaces of degree 5 was considered by Skorobogatov in [S1] (see also [S2]). Recently, Hassett and Tschinkel have investigated the Cox rings and the universal torsors for some interesting special cubic surfaces [H-T].

Remark 1.3. If X is a smooth projective toric variety and $U \subset X$ is the open dense torus orbit, then the choice of a point $p \in U$ defines an isomorphism of U with the algebraic torus T , so that the subgroup $G \subset \mathbb{k}[U]^*$ can be identified with the character group of T . In this way, one can show that $\text{Cox}(X, U, p)$ is isomorphic to a polynomial ring in n variables (n is the number of irreducible components of $X \setminus U$, cf. [Cox]).

Remark 1.4. The field of fractions of the ring $\text{Cox}(X, U, p)$ is a pure transcendental extension of degree r of the field of rational functions on X . Therefore, $\dim \text{Spec } \Gamma(X, U, p) = \dim X + r$, if $\Gamma(X, U, p)$ is a finitely generated \mathbb{k} -algebra.

Let X_r be a smooth Del Pezzo surface obtained from \mathbb{P}^2 by blow-up of $r \leq 8$ points in general position. It is well known that for $r \in \{3, 4, 5, 6, 7, 8\}$ the Picard group $\text{Pic}(X_r)$ contains a canonical root system $R_r \in \{A_2 \times A_1, A_4, D_5, E_6, E_7, E_8\}$. Moreover, the natural embedding $\text{Pic}(X_{r-1}) \hookrightarrow \text{Pic}(X_r)$ induces the inclusion of root systems $R_{r-1} \hookrightarrow R_r$. If $G(R_r)$ is a connected algebraic group corresponding to the root system R_r , then the embedding $R_{r-1} \hookrightarrow R_r$ defines a maximal parabolic subgroup $P(R_{r-1}) \subset G(R_r)$ [Hm]. We expect that for $r \geq 4$ there should be some relation between a Del Pezzo surface X_r and the GIT-quotient of the homogeneous space $G(R_r)/P(R_{r-1})$ modulo the action of a maximal torus T_r of $G(R_r)$.

Our starting observation is the well-known isomorphism $X_4 \cong G(3, 5)/T_4$ which follows from an isomorphism between the homogeneous coordinate ring of the Grassmannian $G(3, 5) = G(A_4)/P(A_2 \times A_1) \subset \mathbb{P}^9$ and the Cox ring of X_4 (see 4.1). Another proof of this fact follows from the identification of X_4 with the moduli space $\overline{M}_{0,5}$ of stable rational curves with 5 marked points [K].

In this paper, we start an investigation of the Cox ring of Del Pezzo surfaces X_r ($r \geq 4$). It is natural to choose the classes of all exceptional curves $E_1, \dots, E_{N_r} \subset X_r$ as a generating set for the Picard group $\text{Pic}(X_r)$. There is a natural $\mathbb{Z}_{\geq 0}$ -grading on $\text{Pic}(X_r)$ defined by the intersection with the anticanonical divisor $-K$.

We prove some general properties of the Cox rings of a Del Pezzo surface X_r ($r \geq 4$) and show their similarity to the homogeneous coordinate ring of $G(R_r)/P(R_{r-1})$. We remark that the homogeneous space $G(R_r)/P(R_{r-1})$ can be interpreted as the orbit of the highest weight vector in some natural irreducible representation of $G(R_r)$.

Remark 1.5. Some other connections between Del Pezzo surfaces and the corresponding algebraic groups were considered also by Friedman and Morgan in [F-M]. A similar topic was considered by Leung in [Le].

In this paper, we show that the Cox ring of a Del Pezzo surface X_r is generated by elements of degree 1. This implies that the homogeneous coordinate ring of $G(R_r)/P(R_{r-1})$ is naturally graded by the monoid of effective divisor classes on the surface X_r (the same monoid defines the multigrading of the Cox ring of X_r). Moreover, we obtain some results of the quadratic relations between the generators of the Cox ring of X_r .

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2. DEL PEZZO SURFACES

Let us summarize briefly some well-known classical results on Del Pezzo surfaces which can be found in [Ma, Dem, Na].

One says that r ($r \leq 8$) points p_1, \dots, p_r in \mathbb{P}^2 are in *general position* if there are no 3 points on a line, no 6 points on a conic ($r \geq 6$) and a cubic having seven points and one of them double does not have the eighth one ($r = 8$).

Denote by X_r ($r \geq 3$) the Del Pezzo surfaces obtained from \mathbb{P}^2 by blowing up of r points p_1, \dots, p_r in general position. If $\pi : X_r \rightarrow \mathbb{P}^2$ the corresponding projective morphism, then the Picard group $\text{Pic}(X_r) \cong \mathbb{Z}^{r+1}$ contains a \mathbb{Z} -basis l_i , ($0 \leq i \leq r$), $l_0 = [\pi^*\mathcal{O}(1)]$ and $l_i := [\pi^{-1}(p_i)]$, $i = 1, \dots, r$. The intersection form $(*, *)$ on $\text{Pic}(X_r)$ is determined in the chosen basis by the diagonal matrix: $(l_0, l_0) = 1$, $(l_i, l_i) = -1$ for $i \geq 1$, $(l_i, l_j) = 0$ for $i \neq j$. The anticanonical class of X_r equals $-K = 3l_0 - l_1 - \dots - l_r$. The number $d := (K, K) = 9 - r$ is called the *degree* of X_r . The anticanonical system $|-K|$ of a Del Pezzo surface X_r is very ample if $r \leq 6$, it determines a two-fold covering of \mathbb{P}^2 if $r = 7$, and it has one base point, determining a rational map to \mathbb{P}^1 if $r = 8$. Smooth rational curves $E \subset X_r$ such that $(E, E) = -1$ and $(E, -K) = 1$ are called *exceptional curves*.

Theorem 2.1. [Ma] *The exceptional curves on X_r are the following:*

- (1) *blown-up points p_1, \dots, p_r ;*
- (2) *lines through pairs of points p_i, p_j ;*
- (3) *conics through 5 points from $\{p_1, \dots, p_r\}$ ($r \geq 5$);*
- (4) *cubics, containing 7 points and 1 of them double ($r \geq 7$);*
- (5) *quartics, containing 8 points and 3 of them double ($r = 8$);*
- (6) *quintics, containing 8 of point and 6 of them double ($r = 8$);*
- (7) *sextics, containing 8 of those points, 7 of them double and 1 triple ($r = 8$).*

The number N_r of exceptional curves on X_r is given by the following table:

r	3	4	5	6	7	8
N_r	6	10	16	27	56	240

The root system $R_r \subset \text{Pic}(X_r)$ is defined as

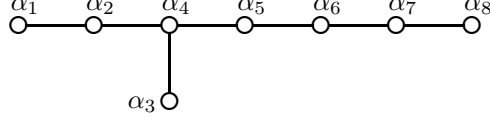
$$R_r := \{\alpha \in \text{Pic}(X_r) : (\alpha, \alpha) = -2, (\alpha, -K) = 0\}.$$

It is easy to show that R_r is exactly the set of all classes $\alpha = [E_i] - [E_j]$ where E_i and E_j are two exceptional curves on X_r such that $E_i \cap E_j = \emptyset$.

The corresponding Weyl group W_r is generated by the reflections $\sigma : x \mapsto x + (x, \alpha)\alpha$ for $\alpha \in R_r$. There are so called *simple roots* $\alpha_1, \dots, \alpha_r$ such that the corresponding reflexions $\sigma_1, \dots, \sigma_r$ form a minimal generating subset of W_r . The set of simple roots can be chosen as

$$\begin{aligned} \alpha_1 &= l_1 - l_2, \alpha_2 = l_2 - l_3, \alpha_3 = l_0 - l_1 - l_2 - l_3, \\ \alpha_i &= l_{i-1} - l_i, \quad i \geq 4. \end{aligned}$$

The blow up morphism $X_r \rightarrow X_{r-1}$ determines an isometric embedding of the Picard lattices $\text{Pic}(X_{r-1}) \hookrightarrow \text{Pic}(X_r)$. This induces the embeddings for root systems, simple roots and Weyl groups W_r . For $r \geq 3$, the Dynkin diagram of R_r can be considered as the subgraph on the vertices α_i ($i \leq r$) of the following graph:



In particular, we obtain $R_3 = A_2 \times A_1, R_4 = A_4, R_5 = D_5, R_6 = E_6, R_7 = E_7, R_8 = E_8$.

Denote by $\varpi_1, \dots, \varpi_r$ the dual basis to the \mathbb{Z} -basis $-\alpha_1, \dots, -\alpha_r$. Each ϖ_i is the highest weight of an irreducible representation of $G(R_r)$ which is called a *fundamental representation*. We shall denote by $V(\varpi)$ the representation space of $G(R_r)$ with the highest weight ϖ .

Definition 2.2. A dominant weight ϖ is called *minuscule* if all weights of $V(\varpi)$ are nonzero and the W_r -orbit of the highest weight vector is a \mathbb{k} -basis of $V(\varpi)$ [G/P-I]. A dominant weight ϖ is called *quasiminuscule* [G/P-III], if all nonzero weights of $V(\varpi)$ have multiplicity 1 and form an W_r -orbit of ϖ (the zero weight of $V(\varpi)$ may have some positive multiplicity).

One can see from the explicit description of the root systems R_r that ϖ_r is minuscule for $3 \leq r \leq 7$, and ϖ_8 is quasiminuscule.

The dimension d_r of the irreducible representation $V(\varpi_r)$ of $G(R_r)$ is given by the following table:

r	4	5	6	7	8
d_r	10	16	27	56	248

We will need the following statement:

Proposition 2.3. *Let D be a divisor on a Del Pezzo surface X_r ($2 \leq r \leq 8$) such that $(D, E) \geq 0$ for every exceptional curve $E \subset X_r$. Then the following statements hold:*

- (i) *the linear system $|D|$ has no base points on any exceptional curve $E \subset X_r$;*
- (ii) *if $r \leq 7$, then the linear system $|D|$ has no base points on X_r at all.*

Proof. Induction on r . If $r = 2$, then there exists exactly 3 exceptional curves E_0, E_1, E_2 , whose classes in the standard basis are $l_0 - l_1 - l_2, l_1, l_2$. Moreover $[E_0], [E_1]$ and $[E_2]$ form a basis of the Picard lattice $\text{Pic}(X_2)$. The dual basis w.r.t. the intersection form is $l_0, l_0 - l_1, l_0 - l_2$. Therefore the above conditions on D imply that

$$[D] = n_0 l_0 + n_1 (l_0 - l_1) + n_2 (l_0 - l_2), \quad n_0, n_1, n_2 \in \mathbb{Z}_{\geq 0}$$

So it is sufficient to check that the linear systems with the classes $l_0, l_0 - l_1, l_0 - l_2$ have no base points. The latter immediately follows from the fact that the first system defines the birational morphism $X_2 \rightarrow \mathbb{P}^2$ contracting E_1 and E_2 , the second and third linear systems define conic bundle fibrations over \mathbb{P}^1 .

For $r > 2$, we consider a second induction on $\deg D = (D, -K)$.

If there is an exceptional curve $E \subset X_r$ with $(D, E) = 0$, then the invertible sheaf $\mathcal{O}(D)$ is the inverse image of an invertible sheaf $\mathcal{O}(D')$ on the Del Pezzo surface X_{r-1} obtained by the contraction of E . Since the pull-back of any exceptional curve on X_{r-1} under the birational morphism $\pi_E : X_r \rightarrow X_{r-1}$ is again an exceptional curve on X_r ,

we obtain that D' satisfy all conditions of the proposition on X_{r-1} . By the induction assumption ($r-1 \leq 7$), $|D'|$ has no base points on X_{r-1} . Therefore, $|D| = |\pi_E^* D'|$ has no base points on X_r .

If there is no exceptional curve $E \subset X_r$ with $(D, E) = 0$, then we denote by m the minimal intersection number (D, E) where E runs over all exceptional curves. Since we have $(E, -K) = 1$ for all exceptional curves, the divisor $D' := D + mK$ has nonnegative intersections with all exceptional curves and there exists an exceptional curve $E \subset X_r$ with $(D', E) = 0$. Since $\deg D' = (D', -K) = (D, -K) - m(K, K) < (D, -K) = \deg D$, by the induction assumption, we obtain that $|D'|$ is base point free. If $r \leq 7$, then the anticanonical linear system $|-K|$ has no base points. Therefore, $|D| = |D' + m(-K)|$ is also base point free. In the case $r = 8$, $|-K|$ does have a base point $p \in X_8$. However, p cannot lie on an exceptional curve E , because the short exact sequence

$$0 \rightarrow H^0(X_8, \mathcal{O}(-K - E)) \rightarrow H^0(X_8, \mathcal{O}(-K)) \rightarrow H^0(E, \mathcal{O}(-K)|_E) \rightarrow 0$$

induces an isomorphism $H^0(X_8, \mathcal{O}(-K)) \cong H^0(E, \mathcal{O}(-K)|_E)$ (since $\deg(-K - E) = 0$ and $H^0(X_8, \mathcal{O}(-K - E)) = 0$). \square

3. GENERATORS OF $\text{Cox}(X_r)$

Let $\{E_1, \dots, E_{N_r}\}$ be the set of all exceptional curves on a Del Pezzo surface X_r . We choose a \mathbb{k} -rational point $p \in U := X_r \setminus (\bigcup_{i=1}^{N_r} E_i)$ and denote the ring $\text{Cox}(X_r, U, p)$ (see 1.1) simply by $\text{Cox}(X_r)$.

The ring

$$\text{Cox}(X_r) = \bigoplus_{[D] \in M_{\text{eff}}(X_r)} \text{Cox}(X_r)^{[D]}$$

is graded by the semigroup $M_{\text{eff}}(X_r) \subset \text{Pic}(X_r)$ of classes $[D]$ of effective divisors D on X_r . There is a coarser grading on $\text{Cox}(X_r)$ given by

$$\text{Cox}(X_r)^d := \bigoplus_{\deg [D] = d} \text{Cox}(X_r)^{[D]},$$

where $\deg [D] := (D, -K)$.

Proposition 3.1. *The graded ring $\text{Cox}(X_3)$ is isomorphic to a polynomial ring in 6 variables $\mathbb{k}[x_1, \dots, x_6]$, where x_i are sections defining all 6 exceptional curves on X_3 .*

Proof. The Del Pezzo surface X_3 is a toric variety which can be described as the blow-up of 3 torus invariant points $(1:0:0)$, $(0:1:0)$ and $(0:0:1)$ in \mathbb{P}^2 . So we can apply a general result of Cox on toric varieties [Cox] (see also 1.3). \square

Theorem 3.2. *For $3 \leq r \leq 8$, the ring $\text{Cox}(X_d)$ is generated by elements of degree 1. If $r \leq 7$, then the generators of $\text{Cox}(X_d)$ are global sections of invertible sheaves defining the exceptional curves. If $r = 8$, then we should add to the above set of generators two linearly independent global sections of the anticanonical sheaf on X_8 .*

Proof. Induction on r . The case $r = 3$ is settled by the previous proposition.

For $r > 3$ we choose an effective divisor D on X_r . We call a section $s \in H^0(X_r, \mathcal{O}(D))$ a *distinguished global section* if its support is contained in the union of exceptional curves

of X_r ($r \leq 7$), or if its support is contained in the union of exceptional curves of X_8 and some anticanonical curves on X_8 . Our purpose is to show that the vector space $H^0(X_r, \mathcal{O}(D))$ is spanned by all distinguished global sections.

This will be proved by induction on $\deg D := (D, -K) > 0$.

We consider several cases:

- If there exists an exceptional curve E such that $(D, E) < 0$, then $H^0(X_r, \mathcal{O}(D)|_E) = 0$ and it follows from the exact sequence

$$H^1(X_r, \mathcal{O}(D)|_E) \rightarrow H^0(X_r, \mathcal{O}(D - E)) \rightarrow H^0(X_r, \mathcal{O}(D)) \rightarrow 0$$

that the multiplication by a non-zero distinguished global section of $\mathcal{O}(E)$ induces an epimorphism $H^0(X_r, \mathcal{O}(D - E)) \rightarrow H^0(X_r, \mathcal{O}(D))$. Since $\deg(D - E) = \deg D - 1$, using the induction assumption for $D' = D - E$, we obtain the required statement for D .

- If there exists an exceptional curve E such that $(D, E) = 0$, then $\mathcal{O}(D)$ is the inverse image of a sheaf $\mathcal{O}(D')$ on the Del Pezzo surface X_{r-1} obtained by the contraction of E . Therefore we have an isomorphism $H^0(X_r, \mathcal{O}(D)) \cong H^0(X_{r-1}, \mathcal{O}(D'))$ and, by the induction assumption for $r - 1$, we obtain the required statement for D , because distinguished global sections of $\mathcal{O}(D')$ lift to distinguished global sections of $\mathcal{O}(D)$.
- If $D = -K$, (or, equivalently, if $(D, E) = 1$ for every exceptional curve E), then $\mathcal{O}(D)|_E$ is isomorphic to $\mathcal{O}_E(1)$ and we have $H^1(X_r, \mathcal{O}(D)|_E) = 0$ together with the exact sequence

$$0 \rightarrow H^0(X_r, \mathcal{O}(D - E)) \rightarrow H^0(X_r, \mathcal{O}(D)) \rightarrow H^0(X_r, \mathcal{O}(D)|_E) \rightarrow 0,$$

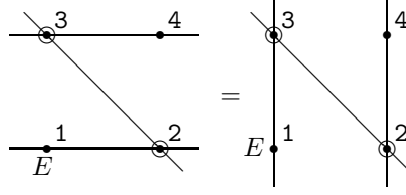
where $H^0(X_r, \mathcal{O}(D)|_E)$ is 2-dimensional. Since $\deg(D - E) = \deg D - 1 < \deg$, we can apply the induction assumption for $D' = D - E$. It remains show that there exists two linearly independent distinguished global sections of $\mathcal{O}(D)$ such that their restriction to E are two linearly independent global sections of $\mathcal{O}(D)|_E$. We describe these two distinguished sections explicitly for each value of $r \in \{4, 5, 6, 7, 8\}$. Without loss of generality we can assume that $[E] = l_1$.

If $r = 4$, then we write the anticanonical class $-K = 3l_0 - l_1 - \dots - l_4$ in the following two ways:

$$\begin{aligned} -K &= (l_0 - l_1 - l_2) + (l_0 - l_3 - l_4) + (l_0 - l_2 - l_3) + l_2 + l_3 \\ &= (l_0 - l_1 - l_3) + (l_0 - l_2 - l_4) + (l_0 - l_2 - l_3) + l_2 + l_3. \end{aligned}$$

These two decompositions of $-K$ determine two distinguished global sections of $\mathcal{O}(-K)$ with support on 5 exceptional curves. The projections of these sections under the morphism $X_4 \rightarrow \mathbb{P}^2$ are shown below in Figure 1.

The restriction of the first section to E vanishes at the intersection point q_1 of E with the exceptional curve with the class $l_0 - l_1 - l_2$. The restriction of the second section to E vanishes at the intersection point q_2 of E with the exceptional curve with the class $l_0 - l_1 - l_3$. It is clear that $q_1 \neq q_2$. So the distinguished anticanonical sections are linearly independent.

FIGURE 1. Two distinguished anticanonical classes for $r = 4$.

If $r = 5$, then we write the anticanonical class as

$$\begin{aligned}
 -K &= 3l_0 - l_1 - \dots - l_5 \\
 &= (l_0 - l_1 - l_2) + (l_0 - l_3 - l_4) + (l_0 - l_4 - l_5) + l_4 \\
 &= (l_0 - l_1 - l_5) + (l_0 - l_2 - l_3) + (l_0 - l_3 - l_4) + l_3.
 \end{aligned}$$

The corresponding distinguished anticanonical sections vanish at two different intersection points of E with the exceptional curves belonging to the classes $l_0 - l_1 - l_2$ and $l_0 - l_1 - l_5$.

If $r = 6$, then we write the anticanonical class as

$$\begin{aligned}
 -K &= 3l_0 - l_1 - \dots - l_6 \\
 &= (l_0 - l_1 - l_2) + (l_0 - l_3 - l_4) + (l_0 - l_5 - l_6) \\
 &= (l_0 - l_1 - l_6) + (l_0 - l_5 - l_4) + (l_0 - l_3 - l_2).
 \end{aligned}$$

The corresponding distinguished anticanonical sections vanish at two different intersection points of E with the exceptional curves belonging to the classes $l_0 - l_1 - l_2$ and $l_0 - l_1 - l_6$.

If $r = 7$, then we write the anticanonical class as

$$\begin{aligned}
 -K &= 3l_0 - l_1 - \dots - l_7 \\
 &= (2l_0 - l_1 - l_2 - l_3 - l_4 - l_5) + (l_0 - l_6 - l_7) \\
 &= (2l_0 - l_7 - l_6 - l_5 - l_4 - l_3) + (l_0 - l_2 - l_1).
 \end{aligned}$$

The corresponding distinguished anticanonical sections vanish at two different intersection points of E with the exceptional curves belonging to the classes $2l_0 - l_1 - l_2 - l_3 - l_4 - l_5$ and $l_0 - l_2 - l_1$.

If $r = 8$, then $\deg D - E = 0$. Therefore, $H^0(X_8, \mathcal{O}(D - E)) = 0$ (see the proof of 2.3) and we have an isomorphism

$$H^0(X_8, \mathcal{O}(D)) \cong H^0(X_8, \mathcal{O}(D)|_E).$$

So $H^0(X_8, \mathcal{O}(D)|_E)$ is generated by the restrictions of the anticanonical sections and we're done.

- If $(D, E) \geq 1$ for all exceptional curves E and $D \neq -K$, then we denote by m the minimum of the numbers (D, E) for all exceptional curves. Let E_0 be an exceptional curve such that $(D, E_0) = m \geq 1$. We define $D' = D - E_0$ and $D'' := D + mK$. By 2.3, $|D'|$ and $|D''|$ have no base points (if $r \leq 7$). In particular, D'' is represented by an effective divisor. Since $\deg D'' = \deg D - m(K, K) < \deg D$, D'' can be seen as zero of a distinguished global section $s \in H^0(X_r, \mathcal{O}(D + mK))$ whose support does not contain the exceptional curve E_0 (if $r \leq 8$). We have the short exact sequence

$$0 \rightarrow H^0(X_r, \mathcal{O}(D')) \rightarrow H^0(X_r, \mathcal{O}(D)) \rightarrow H^0(X_r, \mathcal{O}(D)|_{E_0}) \rightarrow 0.$$

By the induction assumption, the space $H^0(X_r, \mathcal{O}(D'))$ is generated by distinguished global sections. It remains to show that there exist distinguished global sections of $\mathcal{O}(D)$ such that their restriction to E_0 generate the space $H^0(X_r, \mathcal{O}(D)|_{E_0})$. Since $(-mK, E_0) = (D, E_0) = m$, the space $H^0(X_r, \mathcal{O}(D)|_{E_0})$ is isomorphic to $H^0(X_r, \mathcal{O}(-mK)|_{E_0})$. Since $(D'', E_0) = 0$ the distinguished global section $s \in H^0(X_r, \mathcal{O}(D + mK))$ is nonzero at any point of E_0 . Therefore the multiplication by the distinguished global section s defines a homomorphism

$$H^0(X_r, \mathcal{O}(-mK)) \rightarrow H^0(X_r, \mathcal{O}(D))$$

whose restriction to E_0 is an isomorphism

$$H^0(X_r, \mathcal{O}(-mK)|_{E_0}) \cong H^0(X_r, \mathcal{O}(D)|_{E_0}).$$

Therefore, it is enough to show that restrictions of the distinguished global sections of $\mathcal{O}(-mK)$ to E_0 generate the space $H^0(X_r, \mathcal{O}(-mK)|_{E_0})$. Our previous considerations have shown this for $m = 1$. The general case $m \geq 1$ follows now immediately from the fact that the homomorphism $H^0(X_r, \mathcal{O}(-K)) \rightarrow H^0(E_0, \mathcal{O}_{E_0}(1))$ is surjective and the space $H^0(E_0, \mathcal{O}_{E_0}(m))$ is spanned by tensor products of m elements from $H^0(E_0, \mathcal{O}_{E_0}(1))$. \square

Corollary 3.3. *The semigroup $M_{\text{eff}}(X_r) \subset \text{Pic}(X_r)$ of classes of effective divisors on a Del Pezzo surfaces X_r ($2 \leq r \leq X_r$) is generated by elements of degree 1. These elements are exactly the classes of exceptional curves if $r \leq 7$ and the classes of exceptional curves together with the anticanonical class for $r = 8$.*

Proposition 3.4. *If D is an effective divisor of degree ≥ 2 on X_8 , then the vector space $H^0(X_8, \mathcal{O}(D))$ is spanned by distinguished global sections of $\mathcal{O}(D)$ with supports only on exceptional curves.*

Proof. By 3.2 and 3.3, it is sufficient to check the statement for $D = -2K$ and for $D = -K + E$ for any exceptional curve. The latter case immediately follows from 3.2, because $D = -K + E$ is the pull back of the anticanonical sheaf on X_7 obtained by the contraction of E . In the case $D = -2K$, we obtain 120 distinguished global sections of $\mathcal{O}(D)$ from 120 pairs of exceptional curves E_i, E_j such that $(E_i, E_j) = 3$:

$$-2K = 6l_0 - 2l_1 - \dots - 2l_8 = l_1 + (6l_0 - 3l_1 - 2l_2 - \dots - 2l_8).$$

It is well-known (see e.g. [Dem]) that X_8 can be realized as a hypersurface of degree 6 in the weighted projective space $\mathbb{P}(3, 2, 1, 1)$. In particular, the linear system $|-2K|$ defines

a double covering of X_8 over a singular quadratic cone $\mathcal{Q} \cong \mathbb{P}(2, 1, 1) \subset \mathbb{P}^3$. The single singular point $p \in \mathcal{Q}$ is the image of the base-point $b \in X_8$ of $|-K|$ on X_8 . Let $C \subset \mathcal{Q}$ be the ramification locus (C is a curve of degree 6 in $\mathbb{P}(2, 1, 1)$). Then 120 pairs of exceptional curves E_i, E_j on X_8 such that $[E_i] + [E_j] = 2[-K]$ one-to-one correspond to conics in $\mathbb{P}(2, 1, 1)$ which are 3-tangent to the ramification curve C . Since every such conic in \mathcal{Q} is uniquely determined as $\mathcal{Q} \cap H$ for some plane $H \subset \mathbb{P}^3$. Therefore, the distinguished sections in $H^0(X_8, \mathcal{O}(-2K))$ can be identified (up to a scalar multiple) with the above planes $H \subset \mathbb{P}^3$. It remains to show that all these 120 planes H cannot pass through the same common point $x \in \mathbb{P}^3$ for a generic choice of the sextic $C \subset \mathbb{P}(2, 1, 1)$. The later can be checked by standard dimension arguments. \square

Remark 3.5. Since $H^0(X_r, \mathcal{O}(E))$ is 1-dimensional for each exceptional curve $E \subset X_r$, we can choose a nonzero section $x_E \in H^0(X_r, \mathcal{O}(E))$ which is determined up to multiplication by a nonzero scalar. Therefore the affine algebraic variety $\mathbb{A}(X_r) := \text{Spec Cox}(X_r)$ is embedded into the affine space \mathbb{A}^{N_r} on which the maximal torus $T_r \subset G(R_r)$ acts in a canonical way such that the space \mathbb{A}^{N_r} can be identified with the representation space $V(\varpi_r)$ of the algebraic group $G(R_r)$ (if $r \leq 7$). In the case $r = 8$, all 240 exceptional curves on X_8 can be similarly identified with all non-zero weights of the adjoint representation of $G(E_8)$ in $V(\varpi_8)$. The space $V(\varpi_8)$ contains the weight-0 subspace of dimension 8, but the ring $\text{Cox}(X_r)$ has only 2-dimensional space of anticanonical sections. Thus, we cannot identify the degree-1 homogeneous component of $\text{Cox}(X_8)$ with the representation space $V(\varpi_8)$ of $G(E_8)$.

Since the kernel of the surjective homomorphism

$$\deg : \text{Pic}(X_r) \rightarrow \mathbb{Z},$$

can be identified with the character group $\mathfrak{X}(T_r)$ of a maximal torus $T_r \subset G(R_r)$ and the torus T_r acts on the homogeneous space $G(R_r)/P(R_{r-1})$ embedded into the projective space $\mathbb{P}V(\varpi_r)$ we obtain a natural $\text{Pic}(X_r)$ -grading of the homogeneous coordinate ring of the projective variety $G(R_r)/P(R_{r-1})$.

Theorem 3.6. *Let λ be an element in $\text{Pic}(X_r)$. The weight- λ subspace in the homogeneous coordinate ring of the projective variety $G(R_r)/P(R_{r-1})$ is nonzero if and only if λ is represented by an effective divisor on X_r (i.e., $\lambda \in M_{\text{eff}}(X_r)$).*

Proof. It is known that the projective variety $G(R_r)/P(R_{r-1})$ is arithmetically normal and Cohen-Macaulay [D-L, G/P-V]. In particular, the homogeneous coordinate ring of $G(R_r)/P(R_{r-1})$ is generated by elements of degree 1. Therefore, the weight- λ subspace in the coordinate ring is nonzero if and only if λ is a nonnegative integral linear combination of $\text{Pic}(X_r)$ -weights having positive multiplicity in $V(\varpi_r)$. By 3.3 and 3.5, the latter is equivalent to $\lambda \in M_{\text{eff}}(X_r)$.

4. QUADRATIC RELATIONS IN $\text{Cox}(X_r)$

Let us denote $\mathbb{P}(X_r) := \text{Proj Cox}(X_r)$. If $4 \leq r \leq 7$, then the projective variety $\mathbb{P}(X_r)$ is canonically embedded into the projective space \mathbb{P}^{N_r-1} (N_r is the number of exceptional curves on X_r). The affine variety $\mathbb{A}(X_r) \subset \mathbb{A}^{N_r}$ is the affine cone over $\mathbb{P}(X_r)$.

Proposition 4.1. *The ring $\text{Cox}(X_4)$ is isomorphic to the subring of all 3×3 -minors of a generic 3×5 -matrix. In particular, the projective variety $\mathbb{P}(X_4) \subset \mathbb{P}^9$ is isomorphic to the Plücker embedding of the Grassmannian $\text{Gr}(3, 5)$.*

Proof. In order to describe the multiplication in $\text{Cox}(X_4)$, one needs to choose a basis in $\text{Pic}(X_4)$.

Let $x : y : z$ be the homogeneous coordinates on \mathbb{P}^2 . We choose the basis l_0, \dots, l_4 , as in Section 2, i.e., l_0 is the preimage of the line $z = 0$ at infinity, l_1, l_2, l_3, l_4 are classes of the exceptional fibers over 4 points $p_1, \dots, p_4 \in \mathbb{P}^2$. We identify the representatives of each class in $\text{Pic}(X_4)$ with the subsheaves $\mathcal{O}(\sum_{i=0}^4 m_i l_i)$ of the constant sheaf $\mathcal{K}(X_4)$ of rational functions on X_4 . Then the ring multiplication in $\text{Cox}(X_4)$ is just the multiplication of the corresponding rational functions in $\mathcal{K}(X_4)$.

Let $(x_i : y_i : z_i)$ be the coordinates of the blown-up point $p_i \in \mathbb{P}^2$ ($i = 1, \dots, 4$). Consider the 3×5 -matrix

$$M = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 & x/z \\ y_1 & y_2 & y_3 & y_4 & y/z \\ z_1 & z_2 & z_3 & z_4 & 1 \end{pmatrix}.$$

For any 3-element subset $I = \{i, j, k\} \subset \{1, \dots, 5\}$, we denote by M_I the maximal minor of M consisting of the columns with numbers in I taken in the natural order.

We choose the rational functions in $\mathcal{K}(X_4)$ representing the generators x_E of $\text{Cox}(X_4)$ as follows:

$$x_{l_1} := M_{\{2,3,4\}}, \quad x_{l_2} := M_{\{1,3,4\}}, \quad x_{l_3} := M_{\{1,2,4\}}, \quad x_{l_4} := M_{\{1,2,3\}},$$

$$x_{l_0 - l_i - l_j} := M_{\{i,j,5\}}, \quad 1 \leq i < j \leq 4.$$

All these functions are non-zero because the points p_1, \dots, p_4 are in general position.

It is known that the generators of the homogeneous coordinate ring of $G(3, 5)$ are naturally identified with the maximal minors of a generic 3×5 -matrix. Consider the homomorphism φ of the homogeneous coordinate ring of $G(3, 5)$ to $\text{Cox}(X_4)$, which sends these generic minors into the corresponding minors of the matrix M above. Since $\text{Cox}(X_4)$ is generated by $\{x_E\}$, this homomorphism is surjective. By 3.6, φ respects the $\text{Pic}(X_r)$ -grading (in particular, φ respects the $\mathbb{Z}_{\geq 0}$ -grading as well). The surjectivity of φ induces a closed embedding of $\mathbb{P}(X_4)$ into $G(3, 5)$. Since both varieties are irreducible of dimension 6 (see 1.4), we obtain an isomorphism of $\mathbb{P}(X_4)$ and $G(3, 5)$ as subvarieties of \mathbb{P}^9 . Therefore φ is an isomorphism of the homogeneous coordinate ring of $G(3, 5)$ and $\text{Cox}(X_4)$. In particular, $\text{Cox}(X_4)$ is defined by 5 quadratic Plücker relations. One of these relations is

$$M_{\{1,2,5\}} M_{\{3,4,5\}} - M_{\{1,3,5\}} M_{\{2,4,5\}} + M_{\{1,4,5\}} M_{\{2,3,5\}} = 0.$$

□

The article [G/P-I] describes a \mathbb{k} -basis for the homogeneous coordinate ring of G/P in the case, when P is a maximal parabolic subgroup containing a Borel subgroup B such that the fundamental weight ϖ corresponding to P is minuscule (see 2.2). It also shows that this ring is always defined by quadratic relations.

A way to write explicitly the quadratic relations for the orbit of the highest weight vector for any representation of a semisimple Lie group is given in [Li]. A more geometric approach to these quadratic equations is contained in the proof of Theorem 1.1 in [L-T]:

Proposition 4.2. *The orbit G/P_ϖ of the highest weight vector in the projective space $\mathbb{P}V(\varpi)$ is the intersection of the second Veronese embedding of $\mathbb{P}V(\varpi)$ with the subrepresentation $V(2\varpi)$ of the symmetric square $S^2V(\varpi)$. Moreover, these quadratic relations generate the ideal of $G/P_\varpi \subset \mathbb{P}V(\varpi)$.*

We expect that the following general statement is true:

Conjecture 4.3. *The ideal of relations between the degree-1 generators of $\text{Cox}(X_r)$ is generated by quadrics for $4 \leq r \leq 8$.*

For any exceptional curve $E \subset X$, we consider the open chart $U_E \subset \mathbb{P}^{N_r-1}$ defined by the condition $x_E \neq 0$. Thus, we obtain an open covering of $\mathbb{P}(X_r)$ by N_r affine subsets $U_E \cap \mathbb{P}(X_r)$.

Proposition 4.4. *Let X_{r-1} the Del Pezzo surface obtained by the contraction of E on X_r . Then there exist a natural isomorphism*

$$U_E \cap \mathbb{P}(X_r) \cong \mathbb{A}(X_{r-1}).$$

Proof. Let $\pi : X_r \rightarrow X_{r-1}$ be the contraction of E . Then we obtain the ring homomorphism $\pi^* : \text{Cox}(X_{r-1}) \rightarrow \text{Cox}(X_r)$. We shall show that the localization $\text{Cox}(X_r)_{x_E}$ of the ring $\text{Cox}(X_r)$ by the element x_E can be identified with the Laurent polynomial extension of $\pi^* \text{Cox}(X_{r-1})$ by x_E , i.e. there exist a ring isomorphism

$$\text{Cox}(X_r)_{x_E} \cong \pi^* \text{Cox}(X_{r-1})[x_E, x_E^{-1}].$$

For simplicity, we assume that $[E] = l_r$ and $\{l_0, \dots, l_{r-1}\}$ is the pull-back of the standard basis in $\text{Pic}(X_{r-1})$. We remark that any divisor class

$$[D] = m_r l_r + \sum_{i=0}^{r-1} m_i l_i \in \text{Pic}(X_r)$$

is uniquely represented as sum $m_r[E] + [D']$ where $[D'] = \sum_{i=0}^{r-1} m_i l_i \in \pi^*(\text{Pic}(X_{r-1}))$. Using π^* , we identify two fields of rational functions $\mathcal{K}(X_{r-1})$ and $\mathcal{K}(X_r)$. This identification allows us to consider the vector space

$$H^0(X_r, \mathcal{O}(m_r l_r + \sum_{i=0}^{r-1} m_i l_i))$$

as a subspace of

$$H^0(X_{r-1}, \mathcal{O}(\sum_{i=0}^{r-1} m_i l_i)) x_E^{m_r} \subset \pi^* \text{Cox}(X_{r-1})[x_E, x_E^{-1}].$$

For fixed integers m_0, m_1, \dots, m_{r-1} , the embedding of the vector spaces

$$H^0(X_r, \mathcal{O}(m_r l_r + \sum_{i=0}^{r-1} m_i l_i)) \hookrightarrow H^0(X_{r-1}, \mathcal{O}(\sum_{i=0}^{r-1} m_i l_i)) x_E^{m_r}$$

is an isomorphism for sufficiently large m_r . Moreover, this embedding of vector spaces respects the multiplications in $\text{Cox}(X_r)$ and in $\pi^* \text{Cox}(X_{r-1})[x_E, x_E^{-1}]$. Thus, we obtain an embedding of rings

$$\text{Cox}(X_r) \hookrightarrow \pi^* \text{Cox}(X_{r-1})[x_E, x_E^{-1}].$$

On the other hand, it is clear that $\pi^* \text{Cox}(X_{r-1})[x_E, x_E^{-1}]$ is a subring of the localization $\text{Cox}(X_r)_{x_E}$. Thus, we get an isomorphism

$$\text{Cox}(X_r)_{x_E} \cong \pi^* \text{Cox}(X_{r-1})[x_E, x_E^{-1}].$$

Now we remark that the coordinate ring of the affine variety $U_E \cap \mathbb{P}(X_r)$ is degree-0 component of $\text{Cox}(X_r)_{x_E}$. By the above isomorphism, this component is isomorphic to $\text{Cox}(X_{r-1})$. \square

Corollary 4.5. *The singular locus of the algebraic varieties $\mathbb{P}(X_r)$ and $\mathbb{A}(X_r)$ has codimension 7.*

Proof. Since $\mathbb{A}(X_3) \cong \mathbb{A}^6$, we obtain that $\mathbb{P}(X_4)$ is a smooth variety covered by 10 affine charts which are isomorphic to \mathbb{A}^6 . Using the isomorphism $\mathbb{P}(X_4) \cong G(3, 5)$ (see 4.1), we obtain that $\mathbb{A}(X_4)$ has an isolated singularity at 0. Therefore, the singular locus of $\mathbb{P}(X_5)$ consists of 16 isolated points. The singular locus of $\mathbb{P}(X_6)$ is 1-dimensional and the singular locus of $\mathbb{P}(X_7)$ is 2-dimensional etc. \square

Definition 4.6. A divisor class $[D]$ is called *a ruling* if it can be written as a sum of two classes of exceptional curves $[E_i] + [E_j]$ such that $(E_i, E_j) = 1$, or, equivalently, if D satisfies the conditions $(D, D) = 0$, $(D, -K) = 2$. The invertible sheaf corresponding to a ruling determines a conic bundle morphism $X_r \rightarrow \mathbb{P}^1$.

Remark 4.7. Lemma 5.3 of [F-M] says that the Weyl group acts transitively on rulings.

Each ruling $[D]$ can be represented by $r - 1$ different ways as a sum of two classes of exceptional curves corresponding to degenerate fibers of the conic bundle $X_r \rightarrow \mathbb{P}^1$. Thus, we obtain $r - 1$ distinguished sections in the 2-dimensional space $H^0(X_r, \mathcal{O}(D))$. If $r \geq 4$, then for each ruling $[D]$, we obtain in this way $r - 3$ linearly independent quadratic relations between generators of $\text{Cox}(X_r)$.

Remark 4.8. We note that $\text{Pic}(X_4)$ has exactly 5 rulings. Each such a ruling defines a Plücker quadric (see the proof of 4.1).

We cannot expect in general that all quadratic relation among generators are coming from rulings. However, the following statement is true:

Theorem 4.9. *For $4 \leq r \leq 6$, the ring $\text{Cox}(X_r)$ is defined by the radical of the ideal generated by the quadratic relations corresponding to rulings.*

Proof. Let $Z_r \subset \mathbb{A}^{N_r}$ is the affine subvariety defined by the quadratic relations coming from rulings. We want to show that $Z_r = \mathbb{A}(X_r)$ ($4 \leq r \leq 6$).

For $r = 4$, the statement follows from 4.8.

Obviously, the zero $0 \in \mathbb{A}^{N_r}$ is common point of Z_r and $\mathbb{A}(X_r)$ for all r . Consider the affine open coverings of $Z_r \setminus \{0\}$ and $\mathbb{A}(X_r) \setminus \{0\}$ defined by affine open subsets $x_E \neq 0$, where E runs over all exceptional curves of X_r . Using the induction on r and Proposition

4.4, we want to show that $Z_r \cap U_E = \mathbb{A}(X_r) \cap U_E$ for each exceptional curve. For this purpose, it is important to remark that the affine coordinate ring of $Z_r \cap U_E$ is generated by all elements x_F/x_E such that $(E, F) = 0$. For $r = 5, 6$, the last property follows from the fact that if $(E, E') > 0$ for two exceptional curves E, E' on X_r , then $(E, E') = 1$, i.e., $[E] + [E']$ is a ruling and there exists a ruling quadratic relation

$$x_E x_{E'} = \sum_i a_i X_{E_i} X_{E'_i},$$

where all exceptional curves E_i, E'_i do not intersect E . The last property shows that

$$Z_r \cap U_E \cong Z_{r-1} \times (\mathbb{A}^1 \setminus \{0\}).$$

It follows from the proof of 4.4 that

$$\mathbb{A}(X_r) \cap U_E \cong \mathbb{A}(X_{r-1}) \times (\mathbb{A}^1 \setminus \{0\}).$$

By induction, we have the equality $Z_{r-1} = \mathbb{A}(X_{r-1})$. This implies the equality $Z_r \cap U_E = \mathbb{A}(X_r) \cap U_E$ for each exceptional curve. Thus, $Z_r = \mathbb{A}(X_r)$. \square

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